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Date 106 107-2020

Name of College - S.S. College, J-Bad

Dept - Mathematics

Topic - (problem on partial Differentiation) Contd

Class - B.Sc I (HONS)

Date - 06-07-2020.

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Period - I + II.

Problem - If z and u be functions of x and y defined by

$$\{z - g(u)\}^2 = x^2(y^2 - u^2)$$

$$\{z - g(u)\} g'(u) = ux^2$$

Prove that

$$\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = xy$$

Solution → By the question z is a function of x and y therefore

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad \text{——— (I)}$$

But by the question

$$\{z - g(u)\}^2 = x^2(y^2 - u^2) \quad \text{——— (II)}$$

and $$\{z - g(u)\} g'(u) = ux^2 \quad \text{——— (III)}$$

Taking differential of both sides of (2), we get—

$$2\{z-g(u)\}\{dz - g'(u)\,du\} = 2x\,dx\,(y^2-u^2) + x^2(2y\,dy - 2u\,du)$$

$$\Rightarrow \{z-g(u)\}\,dz - \{z-g(u)\}\,g'(u)\,du = x\,dx\,(y^2-u^2) + x^2(y\,dy - u\,du)$$

$$\Rightarrow \{z-g(u)\}\left\{\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right\} - ux^2\,du = x\,dx\,(y^2-u^2) + x^2y\,dy - ux^2\,du.$$

[using (i) and (ii)]

or

$$\{z-g(u)\}\frac{\partial z}{\partial x}dx + \{z-g(u)\}\frac{\partial z}{\partial y}dy = x(y^2-u^2)\,dx + x^2y\,dy.$$

Now equating the co-efficients of dx and dy separately, we get—

$$\{z-g(u)\}\frac{\partial z}{\partial x} = x(y^2-u^2)$$

and $\Rightarrow$

$$\{z-g(u)\}\frac{\partial z}{\partial y} = x^2y.$$

Multiplying these two we get—

$$\{z-g(u)\}^2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = x^3y(y^2-u^2)$$

$$\Rightarrow \cancel{x^2}\cancel{(y^2-u^2)}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = \cancel{x^3}y\cancel{(y^2-u^2)}$$

$$\Rightarrow \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = xy.$$

Hence the result.

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Problem 2-

If $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$ prove that

$$u_x^2+u_y^2+u_z^2=2(xu_x+yu_y+zu_z)$$

Solution — By the question we have

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1 \quad \text{——} (1)$$

Diff. (1) with respect to x partially we have

$$\frac{2x}{a^2+u}-\frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x}-\frac{y^2}{(b^2+u)^2}\frac{\partial u}{\partial x}-\frac{z^2}{(c^2+u)^2}\frac{\partial u}{\partial x}=0$$

$$\Rightarrow \frac{2x}{a^2+u}=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial z} \quad \text{——} (2)$$

Similiarly Diff. (1) partially with respect to y and z successively we get

$$\frac{2y}{b^2+u}=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial y} \quad \text{——} (3)$$

and $$\frac{2z}{c^2+u}=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial z} \quad \text{——} (4)$$

~~Multiplying (2) by x~~
~~(3) by y and (4) by z~~

Now x·2 + y·3 + z·4

$$2\left(\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right)=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)$$

Also Adding the square of (2), (3) and (4)

$$4\left(\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right)^2$$

$$=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]^2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]$$

$$\Rightarrow 4=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right] \quad -(6)$$

Dividing (6) by (5)

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right]$$

i.e. $u_x^2+u_y^2+u_z^2=2[xu_x+yu_y+zu_z]$

Problem 2. If $u=\operatorname{cosec}^{-1}\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$ show that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{12}\left(\frac{13}{12}+\frac{\tan^2 u}{12}\right)$$

Solution :- By the question

$$u=\operatorname{cosec}^{-1}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$$

$$-(5)\quad\Rightarrow \operatorname{cosec} u=\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$$

$$\Rightarrow \sin u = \sqrt{\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$$

$$\Rightarrow \sin u = \sqrt{\frac{x^{1/3}\left[1+(y/x)^{1/3}\right]}{x^{1/2}\left[1+(y/x)^{1/2}\right]}}$$

$$\Rightarrow \sin u = \sqrt{x^{-1/6}\,\frac{1+(y/x)^{1/3}}{1+(y/x)^{1/2}}}$$

$$\Rightarrow \sin u = x^{-1/12}\sqrt{\frac{1+(y/x)^{1/3}}{1+(y/x)^{1/2}}} = v \text{ (say)} \quad \text{(I)}$$

Here v is the homogeneous function of two independent variables x and y and of degree $-1/12$.
Thus By Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = -\frac{1}{2}\sin u \quad \text{(II)}$$

Also $$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = -\frac{1}{12}\left(-\frac{1}{12}-1\right)\sin u \quad \text{(III)}$$

$$\Rightarrow x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = \frac{13}{144}\sin u \quad \text{(III)}$$

Since $v = \sin u$ (From (I))

$$\frac{\partial v}{\partial x} = \frac{\partial \sin u}{\partial u}\cdot\frac{\partial u}{\partial x}$$

$$= \cos u\,\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial \cos u}{\partial u}\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x} + \cos u\,\frac{\partial^2 u}{\partial x^2}$$

$$= -\sin u\left(\frac{\partial u}{\partial x}\right)^2 + \cos u\,\frac{\partial^2 u}{\partial x^2}$$

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Also

$$\frac{\partial^2 v}{\partial x \partial y} = -\sin u \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial x} + \cos u \frac{\partial^2 u}{\partial x \partial y}$$

Again $\frac{\partial v}{\partial y} = \cos u \frac{\partial u}{\partial y}$

and $\frac{\partial^2 v}{\partial y^2} = -\sin u \left(\frac{\partial u}{\partial y}\right)^2 + \cos u \frac{\partial^2 u}{\partial y^2}$

Putting these value in (II) and (III) we get-

From 2, $\cos u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right) = -\frac{1}{12} \sin u$

or $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{12} \tan u$ —— (IV)

From (III)

$$\cos u \left[x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}\right]$$

$$- \sin u \left[x^2 \left(\frac{\partial u}{\partial x}\right)^2 + 2xy \frac{\partial u}{\partial y} \frac{\partial u}{\partial x} + y^2 \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= \frac{13}{144} \sin u$$

or $\cos u \left[x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}\right] - \sin u \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right]^2$

$$= \frac{13}{144} \sin u$$

Which from (IV) is equivalent to

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12} \left(\frac{13}{12} + \frac{\tan^2 u}{12}\right)$$

Result.

Problem If u be a homogeneous function of the nth degree in x, y, z and if $u = f(X, Y, Z)$ where X, Y, Z are the first differential co-efficients of u with regard to x, y, z respectively, prove that-

$$X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y} + Z\frac{\partial f}{\partial Z} = \frac{n}{n-1}u$$

Solution → By the question
u is a homogeneous function of the nth degree in three independent variables x, y and z, therefore,
By Euler's theorem

we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu \longrightarrow (1)$$

Also $u = f(X, Y, Z)$

where $X = \frac{\partial u}{\partial x}$, $Y = \frac{\partial u}{\partial y}$ and $Z = \frac{\partial u}{\partial z}$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial f}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial f}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial f}{\partial Z}\frac{\partial Z}{\partial x} \longrightarrow (2)$$

Again

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial f}{\partial Z}\frac{\partial Z}{\partial y} \longrightarrow (3)$$

Also

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial f}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial f}{\partial Z}\frac{\partial Z}{\partial z} \longrightarrow (4)$$

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Now $x \times (2) + y \times (3) + z \times (4)$

we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

$$= \frac{\partial F}{\partial X}\left[x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right)\right]$$

$$+ \frac{\partial F}{\partial Y}\left[x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right)\right]$$

$$+ \frac{\partial F}{\partial Z}\left[x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)\right]$$

$$\text{or } nu = \frac{\partial F}{\partial X}\left(x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} + z\frac{\partial^2 u}{\partial x \partial z}\right)$$

$$+ \frac{\partial F}{\partial Y}\left(x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + z\frac{\partial^2 u}{\partial y \partial z}\right]$$

$$+ \frac{\partial F}{\partial Z}\left(x\frac{\partial^2 u}{\partial z \partial x} + y\frac{\partial^2 u}{\partial z \partial y} + z\frac{\partial^2 u}{\partial z^2}\right) \quad \text{--- (5)}$$

Differentiating (1) partially with respect to x (keeping y and z constant), we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} + z\frac{\partial^2 u}{\partial x \partial z} = n\frac{\partial u}{\partial x}$$

$$\text{or } x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} + z\frac{\partial^2 u}{\partial x \partial z} = (n-1)\frac{\partial u}{\partial x} = (n-1)X.$$

Similarly differentiating (1) partially with respect to y and z successively, we get

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + z\frac{\partial^2 u}{\partial y \partial z} = (n-1)y$$

$$x\frac{\partial^2 u}{\partial z \partial x} + y\frac{\partial^2 u}{\partial z \partial y} + z\frac{\partial^2 u}{\partial z^2} = (n-1)z$$

Putting all the above values in (1), we get –

$$nu = (n-1)\left[x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}\right]$$

Hence

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = \frac{n}{n-1}u.$$

Proved

End.